Minimum-Energy Pose Filtering on the Special Euclidean Group

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1 Introduction

Obtaining a robust estimate for the pose (attitude and position) of a rigid body moving in three dimensional space using noisy vectorial measurements is a challenging problem. The underlying geometry of pose space, the special Euclidean group SE(3), makes this problem highly nonlinear and sensitive to measurement noise. According to a recent survey [9], most attitude estimation applications in robotics are currently tackled using extended Kalman filter (EKF) based methods, cf. [4, 1, 19]. However, implementing these methods using linearization and sampling techniques that do not respect the underlying geometry of the system's state space may cause convergence and stability issues, cf. [8]. State of the art EKF-type methods such as the multiplicative extended Kalman filter (MEKF) [15] and the invariant extended Kalman filter (IEKF) [6] try to compensate by applying modifications to the EKF equations in order to preserve the geometric structure of the estimates. A recent body of work on the design of nonlinear observers, cf. [18, 5, 20, 11, 12], directly exploits the geometric structure of attitude and pose to achieve guaranteed stability and convergence of estimates. However, these observers mainly use constant gains that need to be pre-tuned depending on the application.

In recent work [16, 13, 14] the authors have designed nonlinear observers with time varying gains (i.e. nonlinear filters) that are posed directly on the geometric spaces of the unit circle S^1 and the rotation group SO(3). The first two works [16, 13] are heuristic and yield bounds on the distance to optimality of their algorithms. The latter work [14] is based on a systematic deterministic minimum-energy filtering approach due to Mortensen [17]. Although Mortensen's approach is for systems defined on a Euclidean space, the authors in [14] extend it to a system defined on S^1. Krener [10] proved that minimum-energy filters achieve exponentially fast convergence under some conditions including the uniform ob-

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servability of the system. Aguiar *et al.* [2] applied minimum-energy filtering to a kinematic model of pose using perspective outputs by embedding the nonlinear geometry of SE(3) in the Euclidean space $\mathbb{R}^{4\times 4}$. The resulting estimates need to be projected back to SE(3) which arguably yields a sub-optimal filter.

In this work we extend Mortensen's deterministic minimum-energy filtering approach to a kinematic model of pose modelled as an element of the special Euclidean group SE(3). We provide the exact form of a minimum-energy observer on SE(3) and show that it depends on the Hessian of a value function of the associated optimal control problem. We give approximate dynamics of the Hessian by a matrix Riccati equation. The overall proposed filter (observer and the Riccati equation) is second order in the sense that it approximates the dynamics of the second order derivate of the value function by neglecting the third order derivative of the value function. Our method avoids any linearization and sampling errors that may occur in EKF-type filters. By working directly on SE(3) we guarantee a global and unique derivation that is not present in any other representation of the pose [7]. The proposed filter considers pose as a single object and yields proper coupling between the rotation and the translation components that may not be the case in filters that are designed independently for each component. We provide a simulation study that shows strong robustness and low convergence error of the proposed filter in the presence of filter initialization error and measurement errors.

The remainder of the paper is presented as follows. Section 2 introduces the proposed minimum energy filtering problem on SE(3). Section 3 briefly describes the filter derivation and summarizes the proposed filter. We demonstrate the performance of the proposed filter by means of simulations in Section 4.

2 Problem Formulation

The following is a model for the kinematics of the pose of a rigid body, and an associated vectorial measurement model, for which we formulate the problem of minimum-energy filtering. Consider

$$\begin{cases} \dot{X}(t) & = X(t)A(t),\ X(0) = X_0, \\ U(t) & = A(t) + (Bv(t))^{\wedge}, \\ y_i(t) & = X(t)^{-1}\mathring{y}_i + [D_i w_i(t)\ ,\ 1]^{\top},\ i = 1, \cdots, n\ , \end{cases} \tag{1}$$

where X is an SE(3)-valued state signal representing the pose of a body-fixed frame, i.e. a frame attached to a moving rigid body, relative to a reference frame, i.e. a frame fixed at a reference point. We use the following matrix representation of pose that is commonly known as homogeneous coordinates. This model preserves the group structure of $\mathrm{SE}(3) \subseteq \mathbb{R}^{4\times 4}$ with the $GL(4)$ operation of matrix multiplication, i.e, $X_1 X_2 \in \mathrm{SE}(3)$, for all $X_1, X_2 \in \mathrm{SE}(3)$.

$$X = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}.$$

Here the rotation R is an element of the rotation group $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^{\top} R = I, \det(R) = 1\}$ where I is the 3 by 3 identity matrix and the translation p is an element of $\mathbb{R}^3$. The signal $A \in \mathfrak{se}(3) \subseteq \mathbb{R}^{4\times4}$ represents the twist of the moving body expressed in the body-fixed frame and it comprises the angular velocity $\Omega_\times \in \mathfrak{so}(3)$ and the translational velocity $V \in \mathbb{R}^3$ of the moving body in the following matrix representation. Note that $\mathfrak{so}(3) = \{\Omega_\times \in \mathbb{R}^{3\times3} \,|\, \Omega_\times = -\Omega_\times^\top\}$. Recall the cross notation $(\cdot)_\times : \mathbb{R}^3 \longrightarrow \mathfrak{so}(3)$ defined as

$$[w_1\; w_2\; w_3]_\times^\top := \begin{pmatrix} 0 & -w_1 & w_2 \\ w_1 & 0 & -w_3 \\ -w_2 & w_3 & 0 \end{pmatrix}, \text{ then } \quad A = \begin{bmatrix} \Omega_\times & V \\ 0 & 0 \end{bmatrix}. \tag{2}$$

Conversely we define $\mathrm{vex}(\cdot) : \mathfrak{so}(3) \longrightarrow \mathbb{R}^3$ by $\mathrm{vex}(\Omega_\times) = \Omega$. The signals $U \in \mathfrak{se}(3)$ and $v \in \mathbb{R}^6$ denote the body-fixed frame measured velocity input and the input measurement error, respectively. The coefficient matrix $B \in \mathbb{R}^{6\times6}$ allows for different weightings for the components of the unknown input measurement error v. We assume that B is full rank and hence that $Q := BB^\top$ is positive definite. The lift-up notation $(\cdot)^\wedge\colon \mathbb{R}^6 \longrightarrow \mathfrak{se}(3)$ is defined as

$$([z_1\; z_2]^\top)^\wedge := \begin{pmatrix} (z_1)_\times & z_2 \\ 0 & 0 \end{pmatrix} \tag{3}$$

where $z_1, z_2 \in \mathbb{R}^3$. Conversely the lift-down notation $(\cdot)^\vee\colon \mathfrak{se}(3) \longrightarrow \mathbb{R}^6$ is defined as $(([z_1\; z_2]^\top)^\wedge)^\vee = [z_1\; z_2]^\top$.The vectors $\mathring{y}_i \in \mathbb{R}^4 = [\underline{\mathring{y}}_i\,,\; 1]^\top$ where $\underline{\mathring{y}}_i \in \diagdown^3$ are known vector directions in the reference frame. The measurements $y_i \in \mathbb{R}^4 = [\underline{y}_i\,,\; 1]^\top$ where $\underline{y}_i$ are measurements of the $\underline{\mathring{y}}_i$ in the body-fixed frame and the signals $w_i \in \mathbb{R}^3$ are the unknown output measurement errors. The coefficient matrix $D_i \in \mathbb{R}^{3\times3}$ allows for different weightings of the components of the output measurement error w_i. Again we assume that D_i is full rank and $M_i := D_i D_i^\top$ is positive definite.

Consider the cost

$$\begin{aligned} &J(t; X_0, v|_{[0,\,t]}, \{w_i|_{[0,\,t]}\}) = \\ &\qquad \frac{1}{2}\int_0^\top \left(\|Bv\|^2_{Q^{-1}} + \sum_i \|D_i w_i\|^2_{M_i^{-1}} \right) d\tau + \frac{1}{2}\|I - X_0\|^2_{K_0^{-1}}. \end{aligned} \tag{4}$$

in which $K_0 \in \mathbb{R}^{3\times3}$ is symmetric positive definite. The cost (4) can be thought of as a measure of the aggregate energy stored in the unknown signals of system (1).

In this paper we consider generalizing Mortensen's minimum-energy filtering [17] to the invariant pose kinematics (1). In other words, given the past measurements $y_i|_{[0\,,\,t]}$ and $U|_{[0\,,\,t]}$ we find the minimum-energy state estimate at the current time t, $\hat{X}(t)$, such that the cost (4) is minimized. In principle this requires postulating a set of unknown signals $(X_0,\; v|_{[0,\,t]},\; \{w_i|_{[0,\,t]}\})$ that are compatible with the measurements $y_i|_{[0\,,\,t]}$ and $U|_{[0\,,\,t]}$ by fulfilling the system equations (1). One can easily find an estimate for the state at time t using the postulated unknowns by integrating the system (1). In general one might find infinitely many combinations of such unknown signals that lead to many different state estimates. However, minimizing (4) yields a triplet $(X_0^*,\; v^*|_{[0,\,t]},\; \{w_i^*|_{[0,\,t]}\})$ that contains minimum collective energy and yields an associated minimum-energy state trajectory $X^*_{[0,\,t]}$. The subscript $[0,\,t]$ indicates that the optimization takes place on the interval $[0,\,t]$. We pick the final optimal state $X^*_{[0,\,t]}(t)$ as our minimum-energy estimate at time t, $\hat{X}(t) := X^*_{[0,\,t]}(t)$. In the following, rather than repeating this optimization process for every time interval $[0,\,t]$ we use Mortensen's approach to find an iterative dynamical equation that updates the minimum-energy estimate $\hat{X}(t)$ as its state value. More details of the method is given in our previous work [14].

Similar to optimal control theory [3], we define a pre-Hamiltonian for this optimization problem. Note that although our optimization problem is carried out over the triplet $(X_0,\, v|_{[0,\,t]},\, \{w_i|_{[0,\,t]}\})$, we can skip optimizing over the $\{w_i|_{[0,\,t]}\}$ if we replace them using the measurements $y_i|_{[0\,,\,t]}$. Also for now we assume that the initial state X_0 is fixed keeping in mind that we later need to optimize the solution over X_0. Hence the problem becomes very similar to an optimal control problem where we need to optimize the following Hamiltonian over v which can be seen as the control parameter.

$$\mathscr{H}^-(X,\mu^\wedge,v,t) := \frac{1}{2}[v^\top v + \sum_i (X^\top \mathring{y}_i - y_i)^\top R_i^{-1}(X^\top \mathring{y}_i - y_i)] - \mu^\top (U^\vee - Bv), \quad (5)$$

where $\mu \in \mathbb{R}^6$ represents a costate variable $\Theta \in \mathfrak{se}^*(3)$ via $\langle \mu^\wedge, \Gamma\rangle = \Theta(\Gamma)$ for all $\Gamma \in \mathfrak{se}(3)$. In the following the identification of $\Theta \in \mathfrak{se}^*(3)$ with $\mu^\wedge \in \mathfrak{se}(3)$ will be used without further reference. Minimizing the pre-Hamiltonian (5) over v yields the optimal Hamiltonian

$$\mathscr{H}(X,\mu^\wedge,t) = \frac{1}{2}[-\mu^\top Q\mu + \sum_i (X^\top \mathring{y}_i - y_i)^\top R_i^{-1}(X^\top \mathring{y}_i - y_i)] - \mu^\top U^\vee. \quad (6)$$

In order to apply the Hamilton-Jacobi-Bellman principle [3] to this problem the following value function is defined

$$V(X,t) := \min_{v|_{[0,\,t]}} J(t; X_0, v|_{[0,\,t]}), \quad (7)$$

where J is the cost (4) and the minimization is constrained by the system equations (1). The Hamilton-Jacobi-Bellman equation is then

$$\mathscr{H}(X, \mathrm{TL}_X^* \nabla_1 V(X,t), t) - \frac{\partial V}{\partial t}(X,t) = 0. \quad (8)$$

From (4) the initial time boundary condition is

$$V(X_0, 0) = \frac{1}{2}\operatorname{trace}\left[(I - X_0)^\top K_0^{-1}(I - X_0)\right]. \quad (9)$$

Up to here we have address the optimal control part of the problem (by minimizing (4) over v) assuming that X_0 is fixed. To solve the original problem we optimize V over X_0. This is equivalent to optimizating V over the final condition $X(t)$

since the initial and final conditions are deterministically coupled by the optimal input $v^*|_{[0\,,\,t]}$. Assuming that the value function is strictly convex, its minimum is characterized by the final condition

$$\nabla_1 V(X,t)|_{X=\hat{X}(t)} = 0. \tag{10}$$

Solving Equation (10) characterizes $\hat{X}(t)$ as the final value of the minimizing argument $X^*_{[0,t]}(t)$. However, this still requires an explicit solution to a potentially infinite dimensional nonlinear control problem and must be repeated at every time t. To overcome this issue we will use Mortensen's approach [17] to derive a recursive solution to this problem.

3 Filter Derivation

In this section we derive a recursive filter by applying Mortensen's approach [17]. Note that the final condition (10) characterizes the solution $\hat{X}(t)$ at the final time t. The final condition (10) is equivalent to

$$\langle \nabla_1 V(X,t), X\Gamma\rangle|_{X=\hat{X}(t)} = (\mathscr{D}_1 V(X,t)\circ X\Gamma)|_{X=\hat{X}(t)} = 0, \quad \text{for all } \Gamma \in \mathfrak{se}(3). \tag{11}$$

In order to get the dynamics of this solution we calculate the total time derivative of the final condition (11). We use the chain rule to calculate the total time derivative. Note that we maintain the nonlinear geometry by using geometric differentiations.

$$\begin{aligned} &\frac{d}{dt}(\mathscr{D}_1 V(X,t)\circ X\Gamma)|_{X=\hat{X}(t)} = \\ &\qquad (\mathscr{D}_1^2 V(X,t)\circ(X\Gamma,\dot{\hat{X}}) + \mathscr{D}_1\frac{\partial V(X,t)}{\partial t}\circ X\Gamma)|_{X=\hat{X}(t)} = 0, \end{aligned} \tag{12}$$

The second order derivative of the value function is related to the Hessian of the value function as an operator acting on a tangent direction. In order to obtain a matrix formulation we represent this by a matrix $K \in \mathbb{R}^{6\times 6}$ such that

$$\mathscr{D}_1^2 V(\hat{X},t)\circ(\hat{X}\Gamma,\dot{\hat{X}}) = \langle \mathrm{Hess}_1 V(\hat{X},t)\circ\dot{\hat{X}},\hat{X}\Gamma\rangle := \langle \hat{X}\left[K(\dot{\hat{X}})^\vee\right]^\wedge, \hat{X}\Gamma\rangle \tag{13}$$

The term $\mathscr{D}_1\frac{\partial V(X,t)}{\partial t}\circ X\Gamma$ can be calculated after replacing the partial derivative from (8). Therefore, denoting $P := K^{-1}$ we obtain the minimum-energy observer equation.

$$\dot{\hat{X}}(t) = \hat{X}\left(U - \left[P\left(\mathbb{P}\sum_i R_i^{-1}(y_i - \hat{X}^{-1}\mathring{y}_i)\mathring{y}_i^\top \hat{X}^{-\top}\right)^\vee\right]^\wedge\right), \tag{14}$$

where $\hat{X}(0) = I$ is calculated from (9) and (10) and $\mathbb{P} : \mathbb{R}^{4\times 4} \to \mathfrak{se}(3)$ denote the orthogonal projection with respect to the Euclidean inner product $\langle\cdot,\cdot\rangle$, i.e., for all

$A \in \mathfrak{se}(3), M \in \mathbb{R}^{4\times 4}$, one has $\langle A, M\rangle = \langle A, \mathbb{P}(M)\rangle = \langle \mathbb{P}(M), A\rangle$. One verifies that for all $M_1 \in \mathbb{R}^{3\times 3}, m_{2,3} \in \mathbb{R}^3, m_4 \in \mathbb{R}$,

$$\mathbb{P}\left(\begin{bmatrix} M_1 & m_2 \\ m_3^\top & m_4 \end{bmatrix}\right) = \begin{bmatrix} \mathbb{P}_a(M_1) & m_2 \\ 0 & 0 \end{bmatrix}.$$

Here, the symmetric projector $\mathbb{P}_s$ is defined by $\mathbb{P}_s(M) := 1/2(M+M^\top)$ for all $M \in \mathbb{R}^{n\times n}$ while the skew-symmetric projector $\mathbb{P}_a$ is defined by $\mathbb{P}_a(M) := 1/2(M - M^\top)$.

Equation (14) depends on the gain P and to implement the observer we need to calculate the dynamics of the gain K and it's inverse P. According to the Mortensen's approach This is done by calculating the total time derivative of the following equation.

$$\langle \dot{K}\gamma, \omega\rangle = \frac{d}{dt}(\mathscr{D}_1^2 V(\hat{X}, t)\circ(\hat{X}\Gamma, \hat{X}\Omega))^\vee, \quad \text{for all } \gamma, \omega \in \mathbb{R}^6, \tag{15}$$

where $\Gamma = \gamma^\wedge$ and $\Omega = \omega^\wedge$. Calculating the right hand side using the chain rule and then using the HJB equation (8), the final condition (11) and neglecting the third order derivatives of the value function we get a Riccati equation for the dynamics of P.

In summary the following filter is obtained.

$$\dot{\hat{X}}(t) = \hat{X}\,(U - (Pl)^\wedge), \tag{16a}$$

$$\dot{P} = Q + 2\mathbb{P}_{sym}(PU) - \mathbb{P}_{sym}(P(Pl)^\wedge) + PEP + PSP. \tag{16b}$$

where

$$l := \left(\mathbb{P}\sum_i R_i^{-1}(y_i - \hat{X}^{-1}\mathring{y}_i)\mathring{y}_i^\top \hat{X}^{-\top}\right)^\vee, \tag{17a}$$

$$E := \begin{pmatrix} \operatorname{trace}(\Delta)I - \Delta & \sum_i M_i^{-1}(\underline{\hat{y}}_i)_\times \\ -\sum_i (\underline{\hat{y}}_i)_\times M_i^{-1} & 0 \end{pmatrix}, \tag{17b}$$

$$\underline{\hat{y}}_i := \hat{R}^\top(\underline{\mathring{y}}_i - \hat{p}),\ \Delta := \sum_i \mathbb{P}_s(M_i^{-1}(\underline{\hat{y}}_i - \underline{y}_i)\underline{\hat{y}}_i^\top), \tag{17c}$$

$$S := \begin{pmatrix} \sum_i (\underline{\hat{y}}_i)_\times M_i^{-1}(\underline{\hat{y}}_i)_\times & -\sum_i (M_i^{-1}(\underline{\hat{y}}_i - \underline{y}_i))_\times \\ \sum_i (M_i^{-1}(\underline{\hat{y}}_i - \underline{y}_i))_\times & -\sum_i M_i^{-1} \end{pmatrix}. \tag{17d}$$

Here $\hat{X}(0) = I$ and $P(0) = \operatorname{diag}(\operatorname{trace}(K_0^{-1})I - K_0^{-1}\ ,\ 0)$ are calculated using (9) and (10). The projection $\mathbb{P}_{sym} : \mathbb{S}^{6\times 6} \times \mathfrak{se}(3) \longrightarrow \mathbb{S}^{6\times 6}$ where $\mathbb{S}^{n\times n}$ is the space of symmetric matrices in $\mathbb{R}^{n\times n}$ is defined as follows. For $P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix}$ and $A = \begin{bmatrix} \Omega_\times & V \\ 0 & 0 \end{bmatrix}$ where $P_1, P_3 \in \mathbb{S}^{3\times 3}$, $P_2 \in \mathbb{R}^{3\times 3}$, $\Omega_\times \in \mathfrak{so}(3)$ and $V \in \mathbb{R}^3$ we define

$$\mathbb{P}_{sym}(PA) := \begin{bmatrix} \mathbb{P}_s(P_1\Omega_\times) & \frac{1}{2}(P_2\Omega_\times - \Omega_\times P_2) \\ \frac{1}{2}(P_2^\top\Omega_\times - \Omega_\times P_2^\top) & \mathbb{P}_s(P_3\Omega_\times) \end{bmatrix}.$$

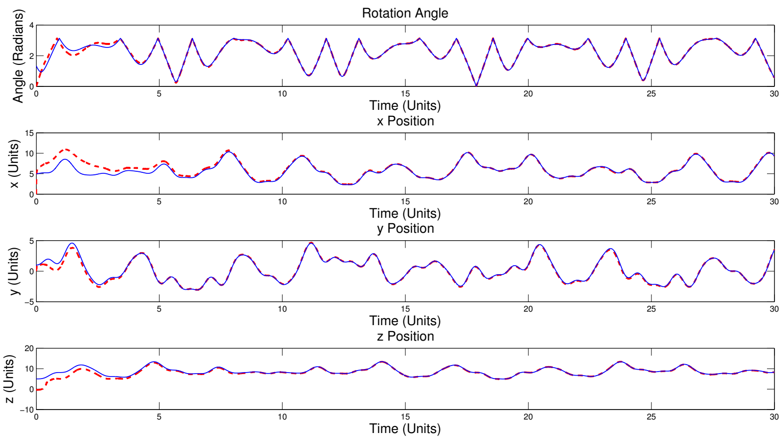

Figure 1: The tracking performance of the proposed filter is shown for the rotation and the translation parts of the system's trajectory. Note that the doted red line is the Filter's trajectory while the solid blue line is the trajectory of the system.

In the next section we present a basic simulation study showing the performance of this filter.

4 Simulations

In this section we present a simulation study of the performance of the proposed filter in the presence of filter initialization and measurement errors. The attitude part of the filter is implemented in unit quaternions similar to the previous work [12]. We consider sinusoidal angular and linear velocity inputs contaminated with measurement errors. Three orthogonal reference directions were considered. The proposed filter was implemented using measurements of these directions contaminated with measurement errors. Figure 1 shows that the proposed filter's trajectory $\hat{X}$ tracks the original system's trajectory X with a fast vanishing transition error and a very small asymptotic error despite the large measurement errors considered.

Bibliography

[1] A. Doucet and N. De Freitas. *Sequential Monte Carlo methods in practice*. Springer, 2001.

[2] A. Aguiar and J. Hespanha. Minimum-energy state estimation for systems with perspective outputs. *IEEE Transactions on Automatic Control*, 51(2):226–241, 2006.

[3] M. Athans and P. Falb. *Optimal control: an introduction to the theory and its applications*. McGraw-Hill, 1966.

[4] B. D. O. Anderson and J. Moore. *Optimal filtering*. Prentice Hall, 1979.

[5] S. Bonnabel, P. Martin, and P. Rouchon. Non-linear symmetry-preserving observers on Lie groups. *IEEE Transactions on Automatic Control*, 54(7):1709–1713, 2009.

[6] Bonnabel, S. and Martin, P. and Salaun, E. Invariant Extended Kalman Filter: theory and application to a velocity-aided attitude estimation problem. In *Proceedings of the 48th IEEE Conference on Decision and Control*, pages 1297 –1304, 2009.

[7] Chaturvedi, N.A. and Sanyal, A.K. and McClamroch, N.H. Rigid-body attitude control. *Control Systems, IEEE*, 31(3):30–51, 2011.

[8] D. Choukroun, I. Bar-Itzhack, and Y. Oshman. A novel quaternion Kalman filter. *IEEE Transactions on Aerospace and Electronic Systems*, 42(1):174–190, 2006.

[9] J. L. Crassidis, F. L. Markley, and Y. Cheng. Survey of nonlinear attitude estimation methods. *Journal of Guidance Control and Dynamics*, 30:12–28, 2007.

[10] A. Krener. The convergence of the minimum energy estimator. In W. Kang, editor, *Lecture Notes in Control and Information Sciences*, pages 187–208. Springer, 2004.

[11] C. Lageman, J. Trumpf, and R. Mahony. Gradient-like observers for invariant dynamics on a Lie group. *IEEE Transactions on Automatic Control*, 55(2):367–377, 2010.

[12] M.-D. Hua, M. Zamani, J. Trumpf, R. Mahony, and T. Hamel. Observer design on the special Euclidean group SE(3). In *Proceedings of the 50th IEEE Conference on Decision and Control*, pages 8169–8175, 2011.

[13] M. Zamani, J. Trumpf, and R. Mahony. Near-Optimal deterministic filtering on the rotation group. *IEEE Transactions on Automatic Control*, 56(6):1411–1414, 2011.

[14] R. M. M. Zamani, J. Trumpf. Minimum-energy filtering on the unit circle. In *Proceedings of the Australian Control Conference (AUCC)*, pages 236–241, 2011.

[15] F. Markley. Attitude error representations for Kalman filtering. *Journal of guidance, control, and dynamics*, 26(2):311–317, 2003.

[16] P. Coote, J. Trumpf, R. Mahony, and J. C. Willems. Near-optimal deterministic filtering on the unit circle. In *Prooceedings of the 48th IEEE Conference on Decision and Control*, pages 5490–5495, 2009.

[17] R. E. Mortensen. Maximum-likelihood recursive nonlinear filtering. *Journal of Optimization Theory and Applications*, 2(6):386–394, 1968.

[18] R. Mahony, T. Hamel, and J. -M. Pflimlin. Nonlinear complementary filters on the special orthogonal group. *IEEE Transactions on Automatic Control*, 53(5):1203–1218, 2008.

[19] S. J. Julier and J. K. Uhlmann. Reduced sigma point filters for the propagation of means and covariances through nonlinear transformations. In *Proceedings of the American Control Conference*, volume 2, pages 887–892, 2002.

[20] J. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira. A nonlinear position and attitude observer on SE (3) using landmark measurements. *Systems & Control Letters*, 59(3-4):155–166, 2010.